Linear Algebra

Notes by Finley Cooper

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1 Vector Spaces

1.1 Definitions

For this lecture course, $\mathbb{F}$ will always be field.

Definition. (Vector Space) A $\mathbb{F}$-*vector space* (or a vector space over $\mathbb{F}$) is an abelian group $(V, +, \mathbf{0})$ equipped with a function

$$\begin{aligned} \mathbb{F} \times V &\to V \\ (\lambda, v) &\to v \end{aligned}$$

which we call scalar multiplication such that $\forall v, w \in V, \forall \lambda, \mu \in \mathbb{F}$
(i) $(\lambda + \mu)v = \lambda v + \mu v$
(ii) $\lambda(v + w) = \lambda v + \lambda w$
(iii) $\lambda(\mu v) = (\lambda\mu)v$
(iv) $1 \cdot v = v \cdot 1 = v$

Remember that $\mathbf{0}$ and 0 are not the same thing. 0 is an element in the field $\mathbb{F}$ and $\mathbf{0}$ is the additive identity in V.

For an example consider $\mathbb{F}^n$ n-dimensional column vectors with entries in $\mathbb{F}$. We also have the example of a vector space $\mathbb{C}^n$ which is a complex vector space, but also a real vector space (taking either $\mathbb{C}$ or $\mathbb{R}$ as the underlying scalar field).

We also can see that $M_{m\times n}(\mathbb{F})$ form a vector space with m rows and n columns.
For any non-empty set X, we denote $\mathbb{F}^X$ as the space of functions from X to $\mathbb{F}$ equipped with operations such that:

$$\begin{aligned} f + g \text{ is given by } (f + g)(x) &= f(x) + g(x) \\ \lambda f \text{ is given by } (\lambda f)(x) &= \lambda f(x) \end{aligned}$$

Proposition. For all $v \in V$ we have that $0 \cdot v = \mathbf{0}$ and $(-1) \cdot v = -v$ where $-v$ denotes the additive inverse of v.

Proof. Trivial.

Definition. (Subspace) A *subspace* of a $\mathbb{F}$-vector space V is a subset $U \subseteq V$ which is a $\mathbb{F}$-vector space itself under the same operations as V. Equivalently, $(U, +)$ is a subgroup of $(V, +)$ and $\forall \lambda \in \mathbb{F}$, $\forall u \in U$ we have that $\lambda u \in U$.

Remark. Axioms (i)-(iv) are always automatically inherited into all subspaces.

Proposition. (Subspace test) Let V be a $\mathbb{F}$-vector space and $U \subseteq V$ then U is a subspace of V if and only if,
(i) U is nonempty.
(ii) $\forall \lambda \in \mathbb{F}$ and $\forall u, w \in U$ we have that $u + \lambda w \in U$.

Proof. If U is a subspace then U satisfies (i) and (ii) since it contains 0 and is closed. Conversely suppose that $U \subseteq V$ satisfies (i) and (ii). Taking $\lambda = -1$ so $\forall u, w \in V$, $u - w \in U$ hence $(U, +)$ is a subgroup of $(V, +)$ by the subgroup test. Finally taking $u = \mathbf{0}$ so we have that $\forall w \in U, \forall \lambda \in \mathbb{F}$ we have that $\lambda w \in U$. So U is a subspace of V. □

We notate U by $U \leq V$.

For some examples

(i)
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = t \right\} \subseteq \mathbb{R}^3,$$
for fixed $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ iff $t = 0$.

(ii) Take $\mathbb{R}^{\mathbb{R}}$ as all the functions from $\mathbb{R}$ to $\mathbb{R}$ then the set of continuous functions is a subspace.

(iii) Also we have that $C^\infty(\mathbb{R})$, the set of infintely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$ and the subspace of continuous functions.

(iv) A further subspace of all of those subspaces is the set of polynomial functions.

Lemma. For $U, W \leq V$ we have that $U \cap W \leq V$.

Proof. We'll use the subspace test. Both U, W are subspaces so they contain $\mathbf{0}$ hence $\mathbf{0} \in U \cap W$ so $U \cap W$ is nonempty. Secondly take $x, y \in U \cap W$ with $\lambda \in \mathbb{F}$. Then $U \leq V$ and $x, y \in U$ so $x + \lambda y \in U$. Similarly with W so $x + \lambda y \in W$ hence we have that $x + \lambda y \in U \cap W$ hence $U \cap W \leq V$ □

Remark. This does not apply for subspaces, in fact from IA Groups, we know it doesn't even hold for the underlying abelian group.

Definition. (Subspace sum) For $U, W \leq V$, the *subspace sum* of U, W is
$$U + W = \{u + w : u \in U, w \in W\}.$$

Lemma. If $U, W \leq V$ then $U + W \leq V$.

Proof. Simple application of the subspace test.

Remark. $U + W$ is the smallest subgroup of U, W in terms of inclusion, i.e. if K is such that $U \subseteq K$ and $W \subseteq K$ then $U + W \subseteq K$.

1.2 Linear maps, isomorphisms, and quotients

Definition. (Linear map) For V, W $\mathbb{F}$-vector spaces. A *linear map* from V to W is a group homomorphism, φ, from $(V, +)$ to $(W, +)$ such that $\forall v \in V$
$$\varphi(\lambda v) = \lambda \varphi(v)$$

Equivalently to show any function $\alpha : V \to W$ is a linear map we just need to show that $\forall u, w \in V$, $\forall \lambda \in \mathbb{F}$ we have

$$\alpha(u + \lambda w) = \alpha(u) + \lambda\alpha(w).$$

For some examples of linear maps

(i) $V = \mathbb{F}^n, W = \mathbb{F}^m$ $A \in M_{m\times n}(\mathbb{F})$. Then let $\alpha : V \to W$ be given by $\alpha(v) = Av$. Then α is linear.

(ii) $\alpha : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by taking the derivative.

(iii) $\alpha : C(\mathbb{R}) \to \mathbb{R}$ defined by taking the integral from 0 to 1.

(iv) X any nonempty set, $x_0 \in X$,

$$\begin{aligned} \alpha : \mathbb{F}^X &\to \mathbb{F} \\ f &\to f(x_0) \end{aligned}$$

(v) For any V, W the identity mapping from V to V is linear and so is the zero map from V to W.

(vi) The composition of two linear maps is linear.

(vii) For a non-example squaring in $\mathbb{R}$ is not linear. Similiarly adding constants is not linear, since linear maps preserve the zero vector.

Definition. (Isomorphism) A linear map $\alpha : V \to W$ is an *isomorphism* if it is bijective. We say that V and W are isomorphic, if there exists an isomorphism from $V \to W$ and denote this by $V \cong W$.

An example is the vector space $V = \mathbb{F}^4$ and $W = M_{2\times 2}(\mathbb{F})$ we can define the map

$$\begin{aligned} \alpha : V &\to W \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &\to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

Then α is an isomorphism.

Proposition. If $\alpha : V \to W$ is an isomorphism then $\alpha^{-1} : W \to V$ is also an isomorphism.

Proof. Clearly α^{-1} is a bijection. We need to prove that α^{-1} is linear. Take $w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$. So we can write $w_i = \alpha(v_i)$ for $i = 1, 2$. Then

$$\alpha^{-1}(w_1 + \lambda w_2) = \alpha^{-1}(\alpha(v_1) + \lambda\alpha(v_2)) = \alpha^{-1}(\alpha(v_1 + \lambda v_2)) = v_1 + \lambda v_2 = \alpha^{-1}(w_1) + \lambda\alpha^{-1}(w_2)$$

. Hence α^{-1} is linear, so α^{-1} is an isomorphism. □

Definition. (Kernal) Let V, W be $\mathbb{F}$-vector spaces. Then the *kernal* of the linear map $\alpha : V \to W$ is

$$\ker(\alpha) = \{v \in V : \alpha(v) = \mathbf{0}_W\} \subseteq V$$

Definition. (Image) Let V, W be $\mathbb{F}$-vector spaces. Then the *image* of a linear map $\alpha : V \to W$ is

$$\operatorname{im}(\alpha) = \{\alpha(v) : v \in V\} \subseteq W$$

Lemma. For a linear map $\alpha : V \to W$ the following hold.

(i) $\ker \alpha \leq V$ and $\operatorname{im} \alpha \leq W$

(ii) α is surjective if and only if $\operatorname{im} \alpha = W$

(iii) α is injective if and only if $\ker \alpha = \{\mathbf{0}_V\}$

Proof. $\mathbf{0}_V + \mathbf{0}_V = \mathbf{0}_V$, so applying α to both sides any using the fact that α is linear gives that $\alpha(\mathbf{0}_V) = \mathbf{0}_W$. So $\ker \alpha$ is nonempty. The rest of the proof is a simple application of the subspace test.
The second statement is immediate from the definition.
For the final statement suppose α injective. Suppose $v \in \ker \alpha$. Then $\alpha(v) = \mathbf{0}_W = \alpha(\mathbf{0}_w)$ so $v = \mathbf{0}_V$ by injectivity. Hence $\ker \alpha$ is trivial. Conversely suppose that $\ker \alpha = \{0_V\}$ Let $u, v \in V$ and suppose that $\alpha(u) = \alpha(v)$. The $\alpha(u - v) = \mathbf{0}_W$, so $u - v \in \ker \alpha$, so $u = v$. □

For V a $\mathbb{F}$-vector space, $W \leq V$ write

$$\frac{V}{W} = \{v + W : v \in V\}$$

as the left cosets of W in V. Recall that two cosets $v + V$ and $u + W$ are the same coset if and only if $v - u \in W$.

Proposition. V/W is an $\mathbb{F}$-vector space under operations

$$\begin{aligned}(u + W) + (v + W) &= (u + v) + W \\ \lambda(v + W) &= (\lambda v) + W\end{aligned}$$

We call V/W the quotient space of V by W.

Proof. The proof is long and requires a lot of vector space axioms so we'll just sketch out the proof.
We check that operations are well-defined, so for $u, \overline{u}, v, \overline{v} \in V$ and $\lambda \in \mathbb{F}$ if

$$u + W = \overline{u} + W, \quad v + W = \overline{v} + W$$

then

$$(u + v) + W = (\overline{u} + \overline{w}) + W$$

and

$$(\lambda u) + W = (\lambda \overline{u}) + W$$

The vector space axioms are inherited from V. □

Proposition. (Quotient map) The function $\pi_W : V \to \frac{V}{W}$ called a *quotient map* is given by

$$\pi_W(v) = v + W$$

is a well-defined, surjective, linear map with $\ker \pi_W = W$.

Proof. Surjectivity is clear. For linearity let $u, v \in V$ and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} \pi_W(u + \lambda v) &= (u + \lambda v) + W \\ &= (u + W) + (\lambda v + W) \\ &= (u + W) + \lambda(v + W) \\ &= \pi_W(u) + \lambda \pi_W(v) \end{aligned}$$

For $v \in V$, we have that $v \in \ker \pi_W \iff \pi_W(v) = \mathbf{0}_{V/W}$. So $v + W = \mathbf{0}_V + W$ so finally $v = v - \mathbf{0}_V \in W$. □

Theorem. (First isomorphism theorem) Let V, W be $\mathbb{F}$-vector spaces and $\alpha : V \to W$ linear. Then there is an isomorphism

$$\overline{\alpha} : \frac{V}{\ker \alpha} \to \operatorname{im} \alpha$$

given by $\overline{\alpha}(v + \ker \alpha) = \alpha(v)$

Proof. For $u, v \in V$,

$$u + K = v = K \iff u - v \in K \iff \alpha(u - v) = \mathbf{0}_W \iff \alpha(u) = \alpha(v) \iff \overline{\alpha}(u + \ker \alpha) = \overline{\alpha}(v + \ker \alpha)$$

The forward direction shows that $\overline{\alpha}$ is well-defined, and the converse shows that $\overline{\alpha}$ is injective. For surjectivity given $w \in \operatorname{im} \alpha$, there exists some $v \in V$ s.t. $w = \alpha(v)$. Then $w = \overline{\alpha}(v + \ker \alpha)$. Finally for linearity given $u, v \in V$, $\lambda \in \mathbb{F}$,

$$\begin{aligned} \overline{\alpha}((u + \ker \alpha) + \lambda(v + \ker \alpha)) &= \overline{\alpha}((u + \lambda v) + \ker \alpha) \\ &= \alpha(u + \lambda v) \\ &= \alpha(u) + \lambda \alpha(v) \\ &= \overline{\alpha}(u + \ker \alpha) + \lambda \overline{\alpha}(v + \ker \alpha) \end{aligned}$$

So $\overline{\alpha}$ is linear hence is an isomorphism □

1.3 Basis

Definition. (Span) Let V be a $\mathbb{F}$-vector space. Then the *span* of some subset $S \subseteq V$ is

$$\langle S \rangle = \left\{ \sum_{s \in S} \lambda_s \cdot s : \lambda_s \in \mathbb{F} \right\}$$

where $\sum$ denotes finite sums. An expression the form above is called a *linear combination* of S.

We say that S *spans* V if $\langle S \rangle = V$

Definition. (Finite-dimensional) For a vector space V we say that it is *finite-dimensional* if there exists a finite spanning set.